THE WEIL REPRESENTATION

1. Preliminaries

Although a large part of the exposition works for more general fields (of particular importance are p-adic local fields), we will stick to the real numbers in this note.

1.1. **Symplectic Space.** A **symplectic space** over $\mathbb{R}$ is a vector space W together with a non-degenerate anti-symmetric bilinear form $\omega \in \wedge^2 W$, the **symplectic form** on W. The **symplectic group** $\mathrm{Sp}(W)$ of W is defined to be the subgroup of $\mathrm{GL}(W)$ of elements that preserves the symplectic form ω.

Remark 1.1.1. *Following the commonly used convention when dealing with Weil representations, we let* $\mathrm{Sp}(W)$ *act on the* ***right*** *of* W*, and view elements in* W *as row vectors.*

Proposition 1.1.2. $\dim W$ *is even, and there exists an* $\mathbb{R}$*-basis*

$$\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$$

of W *such that*

$$\omega(e_i, f_j) = \delta_{ij},\ \omega(e_i, e_j) = \omega(f_i, f_j) = 0.$$

Definition 1.1.3. *For a subspace* $L \subseteq W$*, we define*

$$L^\perp := \{w \in W : \omega(w, l) = 0 \text{ for all } l \in L\}$$

A subspace $L \subseteq W$ *is called* ***Lagrangian*** *if* $L = L^\perp$.

For example, if $\{e_1, \cdots, e_n, f_1, \cdots, f_n\}$ is a standard basis of W, then $X = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$ and $Y = \mathbb{R}f_1 \oplus \cdots \oplus \mathbb{R}f_n$ are Lagrangian subspaces of W.

Proposition 1.1.4. *A Lagrangian subspace of* W *is of dimension* $\frac{1}{2}\dim W$*. Moreover, for any Lagrangian subspace* $X \subseteq W$*, there exists a Lagrangian subspace* $Y \subseteq W$ *such that* $W = X \oplus Y$*.*

A decomposition $W = X \oplus Y$ with X and Y Lagrangian subspaces is called a **polarization** of W. The example above also gives a polarization.

Definition 1.1.5. *The set* $\Lambda(W)$ *of all Lagrangian subspaces in* W *is a compact submanifold of the Grassmannian* $\mathrm{Gr}(n, W)$ *of* n*-dimensional subspaces of* W*, where we suppose* $\dim W = 2n$*.* $\Lambda(W)$ *is called the* ***Lagrangian Grassmannian*** *of* W*.*

Proposition 1.1.6. $\mathrm{Sp}(W)$ *acts transitively on* $\Lambda(W)$*, the stabilizer of a point* $L_0 \in \Lambda(W)$ *is a* ***Siegel parabolic subgroup*** $P(L_0)$ *of* $\mathrm{Sp}(W)$*, so as manifolds there is an isomorphism*

$$\Lambda(W) \cong P(L_0)\backslash \mathrm{Sp}(W)$$

1.2. **Densities on vector spaces.** Let V be a finite-dimensional real vector space. For any $\alpha \in \mathbb{R}$, an α**-density** on V is a map $\rho : \wedge^{top} V \to \mathbb{R}$ such that

$$\rho(\lambda x) = |\lambda|^\alpha \rho(x) \text{ for } \lambda \in \mathbb{R},\, x \in \wedge^{top} V$$

For example, any $\tau \in \wedge^{top} V^*$ defines an α-density $|\tau|^\alpha$ on V by

$$|\tau|^\alpha(x) = |\langle x, \tau\rangle|^\alpha$$

The space of all α-densities on V is denoted $\mathcal{D}^\alpha(V)$. It is a one-dimensional $\mathbb{R}$-vector space. In particular, $\mathcal{D}^1(V)$ is exactly the space of **volume forms** on V. In other words, it consists of all the real multiples of the Lebesgue measure on V, or it consists of all the real Haar measures on V.

Proposition 1.2.1. *There are natural isomorphisms*

$$\mathcal{D}^\alpha(V) \otimes \mathcal{D}^\beta(V) \cong \mathcal{D}^{\alpha+\beta}(V),\, \mathcal{D}^\alpha(V)^* \cong \mathcal{D}^\alpha(V^*) \cong \mathcal{D}^{-\alpha}(V)$$

Proposition 1.2.2. *For vector spaces* $V_1 \subseteq V_2 \subseteq V_3$*, we have a canonical isomorphism*

$$\mathcal{D}^\alpha(V_3/V_1) \cong \mathcal{D}^\alpha(V_3/V_2) \otimes \mathcal{D}^\alpha(V_2/V_1)$$

2. The Schrödinger Representations

In this section we fix a symplectic vector space (W, ω) of dimension $2n$ over $\mathbb{R}$. We will try to use coordinate-free constructions, but will give some formulas in coordinates that are useful for concrete computations.

2.1. **Heisenberg Lie Algebra.** The **Heisenberg Lie algebra** associated to (W, ω) is defined to be the Lie algebra on the vector space $\mathcal{H}(W) = W \oplus \mathbb{R}\mathbf{c}$ with Lie bracket given by

$$[w_1 + t_1\mathbf{c}, w_2 + t_2\mathbf{c}] = \omega(w_1, w_2)\mathbf{c}$$

It is a 2-step nilpotent Lie algebra over $\mathbb{R}$. For any Lagrangian subspace $L \subseteq W$, $\mathcal{H}(L) = L \oplus \mathbb{R}\mathbf{c}$ is a maximal abelian subalgebra of $\mathcal{H}$.

2.2. **Heisenberg Group.** The **Heisenberg group** associated to (W, ω) is defined to be the group sturcture on the set $H(W) = W \oplus \mathbb{R}$ (whose points will be denoted (w, t) for $w \in W, t \in \mathbb{R}$) given by

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\omega(w_1, w_2)) \tag{2.1}$$

Remark 2.2.1. *Why is there an* $\frac{1}{2}$ *in the formula? Indeed it comes from the BCH formula. As the name suggests, the Heisenberg group should be viewed as the Lie group of the Heisenberg Lie algebra, so we consider the formal exponentials* e^X *for* $X \in \mathcal{H}(W)$*. Since the Heisenberg Lie algebra is 2-step nilpotent, the BCH formula takes a rather simple form*

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$$

for $X, Y \in \mathcal{H}(W)$*, here comes the* $\frac{1}{2}$*. This also suggests that the "coordinates" we used when writing an element in* $H(W)$ *as* (w,t) *is actually the coordinate on the Lie algebra. In other words, the element* (w,t) *in* $H(W)$ *should be viewed as the formal exponential* $\exp(w + t\mathbf{c})$.

The symplectic group $\mathrm{Sp}(W)$ acts on $H(W)$ from the right as $(w,t) \cdot g = (w \cdot g, t)$. This action preserves the center $Z = \{(0,t) : t \in \mathbb{R}\}$ of $H(W)$.

For subsequent use, we define a variant of the Heisenberg group, denoted $\overline{H}(W)$, as the set $W \times \mathbb{T}$ (where $\mathbb{T}$ is the set of complex numbers of norm 1) with multiplication

$$(w_1, z_1)(w_2, z_2) = (w_1 + w_2, z_1 z_2 e^{2\pi i \omega(w_1, w_2)})$$

There is an obvious group homomorphism

$$H(W) \to \overline{H}(W),\ (w,t) \mapsto (w, e^{2\pi i z})$$

with kernel isomorphic to $\mathbb{Z}$ in the center $Z \cong \mathbb{R}$.

2.3. **The Schrödinger Representation.** Take a Lagrangian subspace L of W, then $H(L) = L \oplus \mathbb{R} = \{(l,t) \in H(W) : l \in L\}$ is a maximal abelian subgroup of $H(W)$ (actually it is the image under the exponential map of the maximal abelian subalgebra $\mathcal{H}(L)$ in $\mathcal{H}(W)$). For a non-trivial additive character $\psi : \mathbb{R} \to \mathbb{T}$, we define a character ψ_L of $H(L)$ by

$$\psi_L(l,t) = \psi(t)$$

The **Schrödinger representation** of $H(W)$ associated to L and ψ is the unitary representation $(\pi_\psi(L), S_\psi(L)) := \mathrm{Ind}_{H(L)}^{H(W)} \psi_L$ of $H(W)$ induced from the character ψ_L on the subgroup $H(L)$. Concretely, $S_\psi(L)$ consists of functions φ on $H(W)$ that satisfies

(i) $\varphi(ah) = \psi_L(a)\varphi(h)$ for $a \in H(L), h \in H(W)$.
(ii) The function φ, when viewed as a function on $H(L)\backslash H(W)$, is square-integrable. Note that in this case, since $|\varphi|$ is left $H(L)$-invariant, the simplest way to express this condition is that

$$\int_{H(L)\backslash H(W)} |\varphi(\overline{h})|^2 d\overline{h} < \infty$$

for some (hence all) Haar measure $d\overline{h}$ on $H(L)\backslash H(W)$.

And $H(W)$ acts on $S_\psi(L)$ by right translations. The inner product is given by

$$\langle \varphi_1, \varphi_2 \rangle = \int_{H(L)\backslash H(W)} \varphi_1(h)\overline{\varphi_2(h)} d\overline{h}$$

for **some** Haar measure $d\overline{h}$ on $H(L)\backslash H(W)$. If we need to emphasize the choice of the Haar measure, then this space will be denoted $S_\psi(L, d\overline{h})$.

Remark 2.3.1. *Indeed this is cheating a little. The most rigorous way of constructing induced unitary representations is first consider continuous compactly supported functions, define a Hermitian inner product on it, and take completion.*

Let $W = L \oplus L'$ be a polarization, then $H(L)\backslash H(W) \cong L'$. we can take $L^2(L')$ to be the model of the representation $\pi_\psi(L)$, and we record the explicit formulas of the actions of some elements in $H(W)$:

$$\begin{aligned} (\pi_\psi(X)(l,0)\varphi)(x) &= \psi(\omega(l,x))\varphi(x) \\ (\pi_\psi(X)(l',0)\varphi)(x) &= \varphi(l'+x) \\ (\pi_\psi(X)(0,t)\varphi)(x) &= \psi(t)\varphi(x) \end{aligned} \tag{2.2}$$

Remark 2.3.2. *Now we take a basis of W as in Proposition 1.1.2, which identifies W with the standard symplectic structure on $\mathbb{R}^{2n}$, and we take $\psi : \mathbb{R} \to \mathbb{T}$ to be $\psi(t) = e^{2\pi i t}$. The resulting Schrödinger representation is a unitary representation of the Heisenberg group on $L(\mathbb{R}^n)$, whose space of smooth vectors is the space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$. Note that we use the polarization $W = Y \oplus X$ and realize the representation on $L^2(X)$. We can compute the corresponding Lie algebra action of the Heisenberg algebra on $\mathcal{S}(\mathbb{R}^n)$. It turns out that e_k acts as multiplication operators by $2\pi i x_k$, f_k acts as the differential operator $\frac{\partial}{\partial x_k}$, and* **c** *acts as multiplication by $2\pi i$. This is more or less the position and momentum operators in quantum mechanics.*

2.4. Stone-von Neumann Theorem.

It turns out that for different Lagrangian subspaces L, the representations $\pi_\psi(L)$ are irreducible and unitarily equivalent. Actually we can prove something even stronger:

Theorem 2.4.1 (Stone-von Neumann)**.** *For any Lagrangian subspace L, the representation $\pi_\psi(L)$ is irreducible. Moreover, any irreducible unitary representation T of $H(W)$ such that $T(0,t)$ acts as multiplication by $\psi(t)$ is unitarily equivalent to $\pi_\psi(L)$.*

This theorem tells us that the irreducible unitary representations of $H(W)$ are determined by the central character. We will use (π_ψ, S_ψ) to denote "the" irreducible unitary representation of $H(W)$ with central character ψ.

TODO: add a proof of this theorem.

2.5. **Interwiners.** Given Lagrangian subspaces L_1, L_2 of W, by the Stone-von Neumann theorem above, we know that $\pi_\psi(L_1)$ and $\pi_\psi(L_2)$ are unitarily equivalent. In fact we can write down an explicit interwiner $\mathcal{F}_{L_2,L_1}$ from $S_\psi(L_1)$ to $S_\psi(L_2)$ by taking partial Fourier transform:

$$(\mathcal{F}_{L_2,L_1}\varphi)(h) = \int_{L_1\cap L_2\backslash L_2} \varphi(l_2 h)\psi_{L_1}^{-1}(l_2)dl_2 \tag{2.3}$$

This only defines the operator $\mathcal{F}_{L_2,L_1}$ up to a positive scalar, because we do not have any specified Haar measure on $L_1 \cap L_2\backslash L_2$. If we want to emphasize the choice of the Haar measure, we will denote this operator by $\mathcal{F}^\delta_{L_2,L_1}$, where δ is a Haar measure on $L_2/L_1 \cap L_2$. Nevertheless, we know that there exists a unique choice of Haar measures on $L_1 \cap L_2\backslash L_2$ making the operator $\mathcal{F}_{L_2,L_1}$ unitary. In this subsection we will keep track of all the choices of measures, and to clear the ambiguity in the definition of $\mathcal{F}_{L_2,L_1}$.

Let L be a Lagrangian subspace of W. Recall that the inner product on the Hilbert space $S_\psi(L)$ depends on the choice of a Haar measure on $H(L)\backslash H(W) \cong W/L$, which can be canonically identified with the dual space L^* of L by means of ω. Thus for any $e \in \wedge^{top}L - \{0\}$, the corresponding element $|e| \in \mathcal{D}^1(L^*) \cong \mathcal{D}^1(H(L)\backslash H(W))$ gives a Haar measure on $H(L)\backslash H(W)$ (see Section 1.2), and we define $S_\psi(L, |e|)$ to be the Hilbert space $S_\psi(L, |e|)$ with inner product given by integration with respect to the Haar measure $|e|$.

Suppose we are given forms $e_1 \in \wedge^{top}L_1 - \{0\}$ and $e_2 \in \wedge^{top}L_2 - \{0\}$, so that we have Hilbert spaces $S_\psi(L_1, |e_1|)$ and $S_\psi(L_2, |e_2|)$. We are going to find a Haar measure $\delta \in \mathcal{D}^1(L_2/L_1 \cap L_2)$ such that the corresponding operator

$$\mathcal{F}^\delta_{L_2,L_1} : S_\psi(L_1, |e_1|) \to S_\psi(L_2, |e_2|) \tag{2.4}$$

is unitary. To do this, we first recall that the symplectic form ω restricts to a symplectic form ω' on $(L_1 + L_2)/L_1 \cap L_2$, and $(L_1 + L_2)/L_1 \cap L_2 = (L_1/L_1 \cap L_2) \oplus (L_2/L_1 \cap L_2)$ is a polarization of ω'. The next isomorphism will give us the choice of δ.

Lemma 2.5.1. *We have an isomorphism of one-dimensional* $\mathbb{R}$*-vector spaces*

$$\mathcal{D}^1(L_2/L_1 \cap L_2) \cong \mathcal{D}^{\frac{1}{2}}((L_1 + L_2)/L_1 \cap L_2) \otimes \mathcal{D}^{\frac{1}{2}}(W/L_1) \otimes \mathcal{D}^{-\frac{1}{2}}(W/L_2)$$

Proof. For a direct sum decomposition, we always have $\wedge^{top}(V_1 \oplus V_2) \cong \wedge^{top}V_1 \otimes \wedge^{top}V_2$, so we have

$$\begin{aligned}\mathcal{D}^{\frac{1}{2}}((L_1 + L_2)/L_1 \cap L_2) &= \mathcal{D}^{\frac{1}{2}}((L_1/L_1 \cap L_2) \oplus (L_2/L_1 \cap L_2)) \\ &= \mathcal{D}^{\frac{1}{2}}(L_1/L_1 \cap L_2) \otimes \mathcal{D}^{\frac{1}{2}}(L_2/L_1 \cap L_2)\end{aligned}$$

By Proposition 1.2.2, this gives an isomorphism

$$\mathcal{D}^1(W/L_1 \cap L_2) \cong \mathcal{D}^{\frac{1}{2}}((L_1 + L_2)/L_1 \cap L_2) \otimes \mathcal{D}^{\frac{1}{2}}(W/L_1) \otimes \mathcal{D}^{\frac{1}{2}}(W/L_2)$$

Tensor both sides by $\mathcal{D}^{-1}(W/L_2)$ finishes the proof. $\square$

Let $\delta \in \mathcal{D}^1(L_2/L_1 \cap L_2)$ be the image of the element $|\omega'|^{\frac{1}{2}} \otimes |e_1|^{\frac{1}{2}} \otimes |e_2|^{-\frac{1}{2}} \in \mathcal{D}^{\frac{1}{2}}((L_1 + L_2)/L_1 \cap L_2) \otimes \mathcal{D}^{\frac{1}{2}}(W/L_1) \otimes \mathcal{D}^{-\frac{1}{2}}(W/L_2)$ under the above isomorphism.

Theorem 2.5.2. *The operator $\mathcal{F}^{\delta}_{L_2,L_1}$ in (2.4) is a unitary equivalence.*

TODO: add a subsection on the lattice model.

3. Weil Representation

3.1. **The Projective Weil Representation.** As before, let (π_ψ, S_ψ) be "the" irreducible unitary representation of $H(W)$ with central character ψ. Recall that $\mathrm{Sp}(W)$ acts on $H(W)$ from the right preserving the center of $H(W)$, so for any $g \in \mathrm{Sp}(W)$, we can twist the representation π_χ by g, namely we can define a representation π_ψ^g of $H(W)$ by $\pi_\psi^g(h) = \pi_\psi(h \cdot g)$. Since the action of g preserves the center, π_ψ^g also have central character ψ, so by von Neumann's theorem, they are unitarily equivalent, namely there exists a unitary operator $T(g)$ on S_ψ that interwines π_ψ to π_ψ^g. By Schur's lemma, $T(g)$ is well-defined up to a scalar in $\mathbb{T}$, so $g \mapsto T(g)$ gives a projective representation of $\mathrm{Sp}(W)$, called the **Weil representation**.

To clear the ambiguity in the above definition of $T(g)$, we make use of the canonical construction of interwiners in the previous subsection.

Let L be a Lagrangian subspace of W, and take $e \in \wedge^{top} L - \{0\}$, so we have a unitary representation $(\pi_\psi(L), S_\psi(L, |e|))$ constructed in Section 2.3. We will realize the projective Weil representation unambiguously on this Hilbert space. The right action of $\mathrm{Sp}(W)$ on $H(W)$ induces a left action of $\mathrm{Sp}(W)$ on functions on $H(W)$ given by

$$(A(g)\varphi)(h) = \varphi(h \cdot g) \text{ for } g \in \mathrm{Sp}(W),\ h \in H(W),\ \varphi : H(W) \to \mathbb{C}.$$

Clearly $A(g)$ gives a unitary equivalence

$$A(g) : S_\psi(L, |e|) \xrightarrow{\sim} S_\psi(L \cdot g, |e \cdot g|)$$

where $L \cdot g$ is the image of the Lagrangian space L under the right action of g, which is still Lagrangian, and $e \cdot g$ is the image of e under the isomorphism $\wedge^{top} L \to \wedge^{top}(L \cdot g)$ induced by $g : L \to L \cdot g$. We define $T(g) = \mathcal{F}^{\delta(g)}_{L,L\cdot g} \circ A(g)$, which is a unitary operator on $S_\psi(L, |e|)$. The discussion in Section 2.5 shows that $\delta(g) \in \mathcal{D}^1(L/L \cap L \cdot g)$ is given by the image of

$$|\omega'|^{\frac{1}{2}} \otimes |e \cdot g|^{\frac{1}{2}} \otimes |e|^{-\frac{1}{2}}$$

under the isomorphism in Lemma 2.5.1. Note that $\delta(g)$ is independent of the choice of $e \in \wedge^{top} L$, because after a dilation of e by $\lambda \in \mathbb{R}^*$, $|e|^{-\frac{1}{2}}$ differs by the scalar $|\lambda|^{-\frac{1}{2}}$, and $|e \cdot g|^{\frac{1}{2}}$ differs by $|\lambda|^{\frac{1}{2}}$, so they cancels to give 1. This justifies the notation $\delta(g)$. Also this means we can suppress the choice of e from our notations. So we have proved

Theorem 3.1.1. *We have $T(g)^{-1}\pi_\psi(h)T(g) = \pi_\psi(h \cdot g)$ as operators on $S_\psi(L)$, so $g \mapsto T(g)$ is a projective representation of* $\mathrm{Sp}(W)$ *on $S_\psi(L)$.*

3.2. **The Metaplectic Cover of the Symplectic Group.** Since T is a projective representation of $\mathrm{Sp}(W)$, we can define a 2-cocycle $c : \mathrm{Sp}(W) \times \mathrm{Sp}(W) \to \mathbb{T}$ of $\mathrm{Sp}(W)$ by $T(g_1 g_2) = c(g_1, g_2)T(g_1)T(g_2)$. This cocycle gives a central extension of $\mathrm{Sp}(W)$, on which the projective representation lifts to a true representation.We will figure out the central extension in this subsection.

TODO: add the computation of the cocycle via Maslov (Leray) index.

3.3. **Action on Gaussian Functions: $n = 1$ case.** First we consider the simplest case where $W = \mathbb{R}^2$ and $\psi(t) = e^{2\pi i t}$. In this case the (projective) Weil representation of $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ is realized on the space $L^2(\mathbb{R})$, or we will restrict attention to the smooth vectors $\mathcal{S}(\mathbb{R})$ on which the Heisenberg Lie algebra also acts. We record the concrete formulas for this action:

Proposition 3.3.1. *The Weil representation $T : \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to U(L^2(\mathbb{R}))$ is given as follows: for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $\epsilon = \pm 1, \varphi \in L^2(\mathbb{R})$,*

(3.1)
$$(T(g,\epsilon)\varphi)(x) = \begin{cases} \epsilon i^{\frac{1}{2}(1-\mathrm{sgn}(a))}|a|^{\frac{1}{2}} \exp(\pi i a b x^2)\varphi(ax) & c = 0, \\ \epsilon i^{\frac{\mathrm{sgn}(c)}{2}} |c|^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(\pi i c^{-1}(ax^2 - 2xy + dy^2))\varphi(y)dy & c \neq 0. \end{cases}$$

In particular,

$$(T(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, 1)\varphi)(x) = i^{\frac{1-\mathrm{sgn}(a)}{2}}|a|^{\frac{1}{2}}\varphi(ax)$$
$$(T(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, 1)\varphi)(x) = \exp(\pi i b x^2)\varphi(x)$$
$$(T(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, 1)\varphi)(x) = \sqrt{i} \cdot \widehat{\varphi}(x)$$

where $\widehat{\varphi}(x) = \int_{\mathbb{R}} \exp(-2\pi i x y)\varphi(y)dy$ is the Fourier transform of φ.

Let $\mathbb{H} = \{\tau = x + iy : y > 0\}$ be the upper half complex plane. $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by the usual fractional linear transformations.

Definition 3.3.2. *A* ***Gaussian function*** *in $\mathcal{S}(\mathbb{R})$ is a function of the form $E_\tau(t) = \exp(\pi i \tau t^2)$ for $\tau \in \mathbb{H}$.*

The Gaussian function E_τ is the unique solution of the ODE
$$\frac{\partial}{\partial t}\varphi(t) = 2\pi i \tau t \varphi(t)$$
with initial condition $\varphi(0) = 1$. Note that we can identify the Heisenberg Lie algebra $\mathcal{H}(\mathbb{R}^2)$ on $\mathbb{R}^2$ with $\mathbb{R}\mathbf{e} \oplus \mathbb{R}\mathbf{f} \oplus \mathbb{R}\mathbf{c}$ with Lie bracket given by $[\mathbf{e}, \mathbf{f}] = \mathbf{c}$ and $\mathbf{c}$ central. The Schrödinger representation π of the Heisenberg Lie algebra on $\mathcal{S}(\mathbb{R})$ is given by
$$\mathbf{e} \mapsto \frac{\partial}{\partial t},\ \mathbf{f} \mapsto 2\pi i t,\ \mathbf{c} \mapsto 2\pi i$$

So the above ODE can be written as

$$\pi(\mathbf{e} - \tau\mathbf{f})\varphi = 0 \tag{3.2}$$

where $\mathbf{e} - \tau\mathbf{f}$ is viewed as an element in the complexified Heisenberg Lie algebra $\mathcal{H}(\mathbb{R}^2)_{\mathbb{C}}$, on which the $\mathrm{SL}_2(\mathbb{R})$-action extends $\mathbb{C}$-linearly.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we would like to find the ODE satisfied by $T(g)E_\tau$. By (3.2) we have

$$T(g)\pi(\mathbf{e} - \tau\mathbf{f})\varphi = 0$$

namely

$$\pi((\mathbf{e} - \tau\mathbf{f}) \cdot g^{-1})T(g)\varphi = 0$$

(Note that we used the fact that $T(g^{-1})T(g)$ is a constant.) So the ODE satisfied by $T(g)E_\tau$ is

$$\pi((\mathbf{e} - \tau\mathbf{f}) \cdot g^{-1})\varphi = 0 \tag{3.3}$$

Since $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we have

$$(\mathbf{e} - \tau\mathbf{f}) \cdot g^{-1} = ((c\tau + d)\mathbf{e} - (a\tau + b)\mathbf{f})$$

so we can rewrite (3.3) as

$$\pi(\mathbf{e} - \frac{a\tau + b}{c\tau + d}\mathbf{f})\varphi = 0$$

This is exactly the ODE satisfied by $E_{g\cdot\tau}$. So we have proved

Proposition 3.3.3. *For $\tau \in \mathbb{H}$, $g \in \mathrm{SL}_2(\mathbb{R})$, $T(g)E_\tau$ is a scalar multiple of $E_{g\cdot\tau}$. Namely, the projective Weil representation, when restricted to Gaussian functions, recovers the standard $\mathrm{SL}_2(\mathbb{R})$-action on $\mathbb{H}$.*

We are interested in the scalar in this proposition, so let $\widetilde{T}$ be the Weil representation of the metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ on $L^2(\mathbb{R})$, for $\widetilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ let g be the projection in $\mathrm{SL}_2(\mathbb{R})$, then we have

Proposition 3.3.4.

$$\widetilde{T}(\widetilde{g})E_\tau = (c\tau + d)^{-\frac{1}{2}}E_{g\cdot\tau}$$

Proof. Suppose $\widetilde{T}(\widetilde{g})E_\tau = c(g, \tau)E_{g\cdot\tau}$ (clearly the constant only depends on the projection g of $\widetilde{g}$ in $\mathrm{SL}_2(\mathbb{R})$). Since $\widetilde{T}$ is a representation of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, the function c satisfies the cocycle condition

$$c(g_1g_2, \tau) = s(g_1, g_2)c(g_1, g_2\tau)c(g_2, \tau)$$

where $s(g_1, g_2)$ is the cocycle of the central extension $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ over $\mathrm{SL}_2(\mathbb{R})$. Note that one model of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is

$$\widetilde{\mathrm{SL}}_2(\mathbb{R}) = \{(g, \phi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{Hol}(\mathbb{H}) : g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi^2 = c\tau + d\}$$

It turns out that the factor $(c\tau + d)^{-\frac{1}{2}}$ satisfies the same cocycle condition as $c(g, \tau)$, so it suffices to verify the equality on the generators of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, which is done by Proposition 3.3.1. □

References

[1] S. Kudla, *Notes on Local Theta Correspondence.*
[2] G. Lion and M.Vergne, *The Weil Representation, Maslov Index and Theta Series.*